TRANSIENT MAGNETIC FIELD IN A CONDUCTING CYLINDER

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Summary In the paper we determine the transient magnetic field in a conducting cylinder placed in external longitudinal sine-shaped magnetic field using the solution of Bessell equation in cylindrical co-ordinates, and also applying integral Laplace transformation. The resulting equations are the basis for calculation and graphs of space-time distributions, attenuation and diffusion of the magnetic field strength in the cylinder. The resulting equations can be used to describe volume density of the power lost in the cylinder and to determine substitute parameters of the inductor-cylindrical work system.

1. INTRODUCTION

We deal with the case of a conducting cylinder in the longitudinal sinusoidal magnetic field in the induction metal heating. The magnetic field is generated by an inductor to which the power is supplied from a thyristor converter of nominal frequencies ranging from $16\frac{2}{3}$ Hz to 27,12 M·Hz. Average and high frequencies of the magnetic field generated by the inductor require the use of field methods in the description of the electromagnetic field inside the cylindrical charge [1, 2, 3].

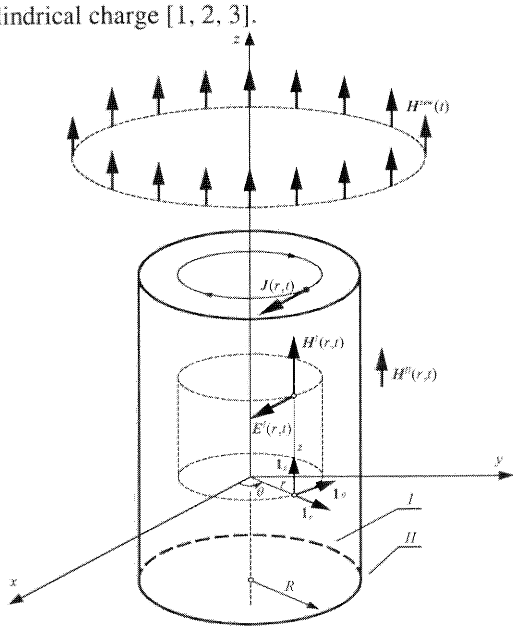

Fig. 1. Conducting cylinder in external longitudinal uniform sinusoidal magnetic field.

Longitudinal sinusoidal magnetic field is external towards the heated cylinder, it has got one component along the z axis (Fig.1) and it is determined with the following formula:

$$\boldsymbol{H}^{ext}(t) = \mathbf{1}_z H_z^{ext}(t), \qquad (1)$$

in which the component of the magnetic field strength along the axis z

$$H_z^{ext}(t) = H_0 \sin(\varpi t + \xi)\mathbf{1}(t), \qquad (1a)$$

where: H_0 - magnetic field amplitude in A·m^{-1},

ω - pulsation in rad·s^{-1},

ξ - initial phase in rad,

$\mathbf{1}(t)$ - Headviside unit step.

2. MAGNETIC FIELD

In the case of an infinitely long conducting cylinder placed in external longitudinal magnetic field (Fig.1) the values describing the electromagnetic field as for the symmetry of the system depend only on the r co-ordinate of the cylindrical co-ordinate system. Then, we deal here with a one-dimensional question with constant magnetic permittivity of the $\mu = \mu_0$ and its constant conductivity γ. As the field $\boldsymbol{H}^{ext}(t)$ has got only one component along the z axis, from the second Maxwell equation $\mathbf{rot}\boldsymbol{E}^{ext}(r,t) = -\mu \frac{\partial \boldsymbol{H}^{ext}(t)}{\partial t}$, the electric field strength has also got one component along the axis Θ, i.e. $\boldsymbol{E}^{ext}(r,t) = -\mathbf{1}_\Theta E_\Theta^{ext}(r,t)$. So we have to deal with a question of the cylindrical wave cast on the lateral surface of the conducting cylinder.

In the general case of a conductor of a chosen kind placed in alternating electromagnetic field some currents are bound to appear, as the total electric field cannot equal zero everywhere in the whole conductor. Those currents are called Foucault currents [4, 5] and are determined by the current density vector $\boldsymbol{J}(r,t)$ - Fig.1. These currents generate the so-called return interaction magnetic field $\boldsymbol{H}^{oz}(r,t)$, which, in the system we are considering, has got one component along the z axis, thus $\boldsymbol{H}^{oz}(r,t) = \mathbf{1}_z H_z^{oz}(r,t)$. In papers [4, 5] it has been shown that this field equals zero. The zero value of the return interaction magnetic field in $r > R_2$ area results form the fact that the lines of the density of current $\underline{J}(r)$ induced in the tubular charge are concentric circles of Oz axis– Fig.1. Then they do not generate any magnetic field outside the tube, as it is also the case with the current in the infinitely long solenoid. Then the total magnetic field in the considered area

$$\boldsymbol{H}^{II}(t)=\mathbf{1}_z H_z^{II}(t)=\mathbf{1}_z H_z^{ext}(t)=\mathbf{1}_z \mathrm{Im}\{\underline{H}_z^{ext}(t)\}, \qquad (2)$$

where

$$\underline{H}_z^{ext}(t)=\underline{H}_0\,\mathrm{e}^{-\eta t}\,\mathrm{e}^{\mathrm{j}\varpi t}\,\mathbf{1}(t), \qquad (2a)$$

where the complex amplitude of the external magnetic field

$$\underline{H}_0 = H_0\,\mathrm{e}^{\mathrm{j}\xi}. \qquad (2b)$$

The required magnetic field strength $H_z^I(r,t)$ in the area of I $(0 \le r \le R)$ is written as $H_z^I(r,t)=\mathrm{Im}\{\underline{H}_z^I(r,t)\}$, where $\underline{H}_z^I(r,t)$ is the complex magnetic field function of real variables r and t. This function fulfils the scalar wave equation in cylindrical co-ordinates [6]

$$\frac{\partial^2 \underline{H}_z^I(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial \underline{H}_z^I(r,t)}{\partial r}-\mu\gamma\frac{\partial \underline{H}_z^I(r,t)}{\partial t}=0. \qquad (3)$$

For $r=R$ it has to be the case of the continuity of the magnetic field strength, i.e. we have the following boundary condition for the complex values:

$$\underline{H}_z^I(R,t)=\underline{H}_z^{ext}(t). \qquad (3a)$$

Moreover we assume a zero initial condition, i.e. for $t=0$

$$\underline{H}_z^I(r,0)=0. \qquad (3b)$$

We solve equation (3) with the boundary condition (3a) applying Laplace integral transform. In order to do this let us denote by $\overline{H}_z^I(r,s)$ the Laplace transform of complex function $\underline{H}_z^I(R,t)$ in relation to variable t, and then we perform the Laplace transformation of the following terms of the differential equation (3), taking into account the zero initial conditions $\underline{H}_z^I(r,0)=0$. Thus, we obtain the following equation [4, 5]:

$$\frac{\partial^2 \overline{H}_z^I(r,s)}{\partial r^2}+\frac{1}{r}\frac{\partial \overline{H}_z^I(r,s)}{\partial r}-s\mu\gamma\,\overline{H}_z^I(r,s)=0 \qquad (4)$$

with the boundary condition

$$\overline{H}_z^I(R,s)=\underline{H}_0\frac{1}{s-s_0}, \qquad (4a)$$

where

$$s_0=\mathrm{j}\varpi. \qquad (4b)$$

Equation (4) is the Bessel equation of zero order of variable r, whose solution is the function [4, 5]

$$\overline{H}_z^I(r,s)=\underline{H}_0\frac{I_0(\sqrt{s}\sqrt{\mu\gamma}r)}{(s-s_0)\,I_0(\sqrt{s}\sqrt{\mu\gamma}R)}, \qquad (5)$$

where $I_0(\sqrt{s}\sqrt{\mu\gamma}r)$ is the modified Bessel function of the complex variable $\sqrt{s}\sqrt{\mu\gamma}r$ of first kind and zero order. The function zeros of the denominator in formula (5) are $s=s_0$ and

$$s=s_k=-\sigma_k=-\frac{x_k^2}{\mu\,\gamma\,R^2}<0, \qquad (5a)$$

where [7]

$$x_k \cong \varphi_k+\frac{1}{8\varphi_k}-\frac{124}{3(8\varphi_k)^3}+\frac{120928}{158(8\varphi_k)^5}-\ldots, \qquad (5b)$$

where

$$\varphi_k=(k-\frac{1}{4})\pi, \quad (k=1,2,3,\ldots). \qquad (5c)$$

Then to calculate the original $\underline{H}_z^I(r,t)$ of the operational function $\overline{H}_z^I(r,s)$ we use the distribution theorem, obtaining [4, 5]

$$\underline{H}_z^I(r,t)=\left[\underline{H}_{z,0}^I(r,t)+\sum_{k=1}^{\infty}\underline{H}_{z,k}^I(r,t)\right]\mathbf{1}(t), \qquad (6)$$

where $\underline{H}_{z,0}^I(r,t)$ is the original of function (5) in the pole $s=s_0$ ($k=0$), while $\underline{H}_{z,k}^I(r,t)$ is the original of this function in the pole $s=s_k$ ($k=1, 2, 3,\ldots$). These originals are given by the following formulas:

$$\underline{H}_{z,0}^I(r,t)=\underline{H}_0\frac{I_0(\underline{\Gamma}\,r)}{I_0(\underline{\Gamma}\,R)}\mathrm{e}^{\mathrm{j}\varpi t}, \qquad (6a)$$

and

$$\underline{H}_{z,k}^I(r,t)=\underline{H}_0\frac{I_0(-\mathrm{j}x_k\frac{r}{R})}{\underline{A}_k(x_k)I_1(-\mathrm{j}x_k)}\exp[-\frac{x_k^2}{\mu\gamma R^2}t], \qquad (6b)$$

where $I_1(-\mathrm{j}\,x_k)$ is the modified Bessel function of first kind and first order,

$$\underline{A}_k(x_k) = \frac{1}{2x_k}[R^2\varpi\mu\gamma - \mathrm{j}\, x_k^2] = \\ = A_k(x_k)\exp[\mathrm{j}\ \alpha_k(x_k)] \quad (6c)$$

and the complex propagation constant

$$\underline{\Gamma} = \sqrt{\mathrm{j}}\sqrt{\varpi\,\mu\,\gamma} = k + \mathrm{j}k = \sqrt{2\mathrm{j}}\,k, \quad (6d)$$

where the coefficient

$$k = \sqrt{\frac{\varpi\,\mu\,\gamma}{2}}, \quad (6e)$$

whose inverse is the depth of the diffusion of the wave inside the well-conducting medium and it is

$$\delta = \frac{1}{k} = \sqrt{\frac{2}{\varpi\,\mu\,\gamma}}. \quad (6f)$$

The exponential form of the Bessel function appearing in formulas (6a) and (6b)

$$\left.\begin{aligned} &I_0(\underline{\Gamma} r) = M_0(\underline{\Gamma} r)\exp[\mathrm{j}\beta_0(\underline{\Gamma} r)],\\ &I_0(\underline{\Gamma} R) = M_0(\underline{\Gamma} R)\exp[\mathrm{j}\beta_0(\underline{\Gamma} R)],\\ &I_0(\text{-j}x_k\frac{r}{R}) = M_{0,k}(\text{-j}x_k\frac{r}{R})\exp[\mathrm{j}\beta_{0,k}(\text{-j}x_k\frac{r}{R})],\\ &I_1(\text{-j}x_k) = M_{1,k}(\text{-j}x_k)\exp[\mathrm{j}\beta_{1,k}(\text{-j}x_k)]. \end{aligned}\right\} \quad (7)$$

lets us to write the functions (6a) and (6b) in real forms, i.e. as real functions of variable r of the cylindrical co-ordinate system and of time t. We obtain respectively

$$H^{I}_{z,0}(r,t) = \mathrm{Im}\{\underline{H}^{I}_{z,0}(r,t)\} = H_0\frac{M_0(\underline{\Gamma} r)}{M_0(\underline{\Gamma} R)}\cdot \\ \cdot\sin[\varpi t + \beta_0(\underline{\Gamma} r) - \beta_0(\underline{\Gamma} R) + \xi] \quad (8)$$

and

$$H^{I}_{z,k}(r,t) = \mathrm{Im}\{\underline{H}^{I}_{z,k}(r,t)\} = \\ H_0\frac{M_0(\text{-j}x_k\frac{r}{R})}{A_k(x_k)\,M_1(\text{-j}x_k)}\exp[-\frac{x_k^2}{\mu\,\gamma\,R^2}t]\cdot \\ \cdot\sin[\beta_{0,k}(\text{-j}x_k\frac{r}{R}) - \beta_{1,k}(\text{-j}x_k) - \alpha_k(x_k) + \xi] \quad (8a)$$

where the module of the complex number $\underline{A}_k(x_k)$

$$A_k(x_k) = \frac{1}{2x_k}\sqrt{(R^2\varpi\mu\gamma)^2 + x_k^4} \quad (8b)$$

and its argument

$$\alpha_k(x_k) = -\mathrm{arc\ tg}\frac{x_k^2}{R^2\varpi\mu\gamma}. \quad (8c)$$

Finally the magnetic field strength in a conducting cylinder placed in external longitudinal magnetic field of a character of an attenuated sinusoid has the following form

$$H^{I}_{z}(r,t) = \left[H^{I}_{z,0}(r,t) + \sum_{k=1}^{\infty} H^{I}_{z,k}(r,t)\right]\mathbf{1}(t). \quad (8d)$$

3. DISTRIBUTION OF THE MAGNETIC FIELD

To work out the graphical presentation of the magnetic field distribution in a conducting cylindrical charge we introduce the variable x corresponding to the variable r of the cylindrical co-ordinate system, as

$$x = \frac{r}{R}, \qquad 0 \le x \le 1, \quad (9)$$

The frequency of the sinusoidal external magnetic field and the conductivity of the charge with regard to its external radius are taken into account through the coefficient

$$\alpha = \frac{R}{\delta} = k\,R. \quad (9a)$$

Then, we obtain:

$$\underline{\Gamma}\ r = \sqrt{2\,\mathrm{j}}\ k\ r = \sqrt{2\,\mathrm{j}}\ \alpha\ x, \quad (9b)$$

$$\underline{\Gamma} R = \sqrt{2\mathrm{j}}\,k\,R = \sqrt{2\mathrm{j}}\,\alpha. \quad (9c)$$

Then we describe the magnetic field in relative units by means of the coefficient

$$h(x,t) = \frac{H^{I}_{z}(x,t)}{H_0} = \\ = \left[h_0(x,t) + \sum_{k=1}^{\infty} h_k(x,t)\right]\mathbf{1}(t) \quad (10)$$

where the fixed component of the magnetic field

$$h_0(x,t) = \frac{M_0(\sqrt{2\mathrm{j}}\,\alpha x)}{M_0(\sqrt{2\mathrm{j}}\,\alpha)}\cdot \\ \cdot\sin[\varpi t + \beta_0(\sqrt{2\mathrm{j}}\,\alpha x) - \beta_0(\sqrt{2\mathrm{j}}\,\alpha) + \xi], \quad (10a)$$

and its transient component

$$h_k(x,t) = \frac{M_0(\text{-j}x_k x)}{A_k(x_k)\,M_1(\text{-j}x_k)} \exp[-\frac{\varpi x_k^2}{2\alpha^2} t] \cdot \tag{10b}$$
$$\cdot \sin[\beta_{0,k}(\text{-j}x_k x) - \beta_{1,k}(\text{-j}x_k) - \alpha_k(x_k) + \xi]$$

The influence of the parameter α on the magnetic field distribution in the cylindrical charge is shown in Fig.2 at $t = T/4$, that is to say for the instantaneous value of the external magnetic field equal to its amplitude. The space-time distribution of this field is shown in Fig.3.

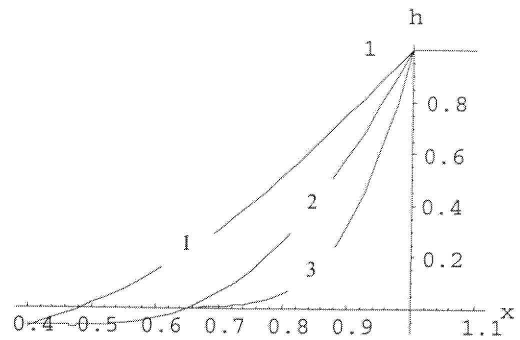

Fig. 2. Diffusion and attenuation of the magnetic field for $t = T/4$, $\omega = \pi \cdot 10^4$ rad·s^{-1}, $\gamma = 58 \cdot 10^6$ S·m^{-1}, $\xi = 0$: $1 - \alpha = 3$, $2 - \alpha = 5$, $3 - \alpha = 10$.

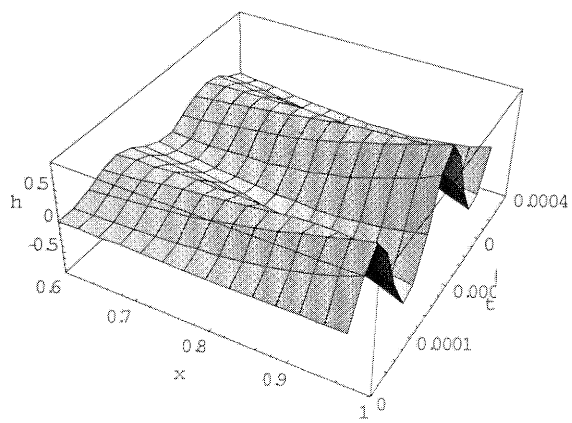

Fig. 3. Time-space distribution of magnetic field strength in a conducting cylinder placed in external longitudinal uniform sinusoidal magnetic field; $\alpha = 5$, $\xi = 0$, $\omega = \pi \cdot 10^4$ rad·s^{-1}, $\gamma = 58 \cdot 10^6$ S·m^{-1}.

4. CONCLUSION

If we do not take into account the transient, i.e. if we consider the conducting cylinder in external longitudinal sinusoidal magnetic field the transient component will disappear and there will only be the fixed component left. The complex magnetic field is then defined with the formula

$$\underline{H}_z^I(r,t) = \underline{H}_{z0}^I(r,t) = \underline{H}_z^I(r)\,\mathrm{e}^{\mathrm{j}\varpi t}, \tag{11}$$

thus, from equation (6a) we obtain

$$\underline{H}_z^I(r) = \underline{H}_0 \frac{I_0(\sqrt{\mathrm{j}}\, m\, r)}{I_0(\sqrt{\mathrm{j}}\, m\, R)}, \tag{11a}$$

which is the solution (given in [6], p. 199, formula (9.109)).

The fact that the transient component appears results in the 'smoothing' of the magnetic field distribution. It is shown in Fig.4.

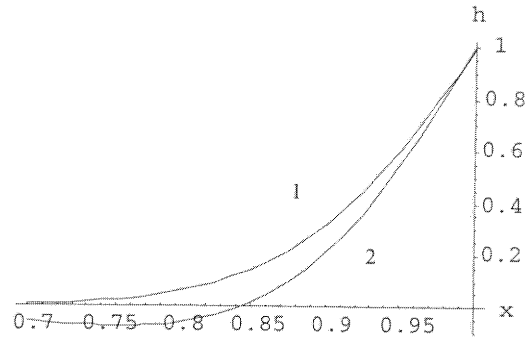

Fig. 4. Distribution of the magnetic field at $t = T/4$, $\xi = 0$, $\omega = \pi \cdot 10^4$ rad·s^{-1}, $\gamma = 58 \cdot 10^6$ S·m^{-1}, $\alpha = 10$: $1 - h(x,t)$ $2 - h_0(x,t)$.

Having determined the magnetic field strength $\boldsymbol{H}^I(r,t) = \mathbf{1}_z H_z^I(r,t)$ we can, form the first Maxwell equation $\mathbf{rot}\boldsymbol{H}^I(r,t) = \boldsymbol{J}(r,t)$, determine the current density $\boldsymbol{J}(r,t) = \mathbf{1}_\Theta J_\Theta(r,t)$. Then also the Poynting vector $\boldsymbol{P}(r,t) = \boldsymbol{E}(r,t) \times \boldsymbol{H}(r,t) = -P_r(r,t)\mathbf{1}_r$, that will let us determine the heating of the cylindrical charge in the case of the transient magnetic field.

REFERENCES

[1] Hering M.: *Fundamentals of Electroheat. Part II* (in Polish). WNT, Warszawa 1998.

[2] Sajdak C., Semek E.: *Induction heating* (in Polish). Wyd. Śląsk, Katowice 1987.

[3] Kurbiel A., Waradzyn Z.: *A Method for Determining Electric Quantities in a Workpiece Heated by Induction.* Electrical Power Quality and Utilisation, Vol. IX, No 1, 2003, ss. 103-108.

[4] Piątek Z, Baron B.: *Longitudinal Magnetic Field of a Character of an Attenuated Sinusoid in a Conducting Cylinder.* Moderní směry výuky elektrotechniky a elektroniky – STO-8, Brno, Czech Republic, 25.-26. září 2002, pp. 141-148.

[5] Piątek Z, Baron B.: *Diffusion of Longitudinal Magnetic Field of a Character of an Attenuated Sinusoid into a Conducting Cylinder.* IV-th International Workshop "Computational Problems of Electrical Engineering", Zakopane, Poland, September 2-5, 2002, pp. 117-122.

[6] Krakowski M.: *Theoretical electrotechnics. Vol. 2. Electromagnetic fields* (in Polish). Wyd. Nauk. PWN, Warszawa 1995.

[7] McLachlan N. W.: *Bessel Functions for Engineers* (in Polish). PWN, Warszawa 1964.